Term Paper
Mathematics 6350 - "Dynamical Systems and Chaos"

Jozo Acksteiner

Testing Chaos and Nonlinearities in T-Bill Rates

- referring to the article in the "FINANCIAL ANALYSTS JOURNAL/ September-October 1991" written by Maurice Larrain

The determination and explanation of changing in asset prices today are rather unsatisfactory - most approaches use the "random walk" hypothesis that means that past prices do not help predicting future prices.
The author of the article, Maurice Larrain, uses a different approach:
For the determination of Treasure-Bill rates he uses a model that combines known "behavioral elements" with elements of nonlinear dynamics.
This model provides an amazingly good approximation of the Treasure-Bill Rates and explains the seemingly randomness of changes.

Fundamentals

There are two models for determination of rates/prices in financial environments:

1) Purely fundamental models, which try to predict future rates/prices r_t at a time t by considering behavioral variables Z such as real income y, nominal money supply M, inflation P and so on.
An equation for this model would be of the form

$r_t = g(Z)$, where $Z = (y,M,P, ...)$. (1)

In case of Treasure Bill Rates income growth or increased inflation would increase money demand and therefore would increase interest rates. Increases in the money supply (except inflation) would lead to a decrease of interest rates.
In this model, r_t is not dependent on past rates/prices r_{t-n} but on fundamental economic determinants.

2) Purely technical models neglect the fundamental economic determinants and use only past rates/prices for determining the new rates/prices.
Such a technical model can be expressed in the following equation:

$r_t = f(r_{t-1}, \ldots, r_{t-n})$. (2)

The random walk hypothesis states that these technical models are not good in predicting future rates/prices, because the relation between past and future prices changes randomly. In the example of Treasure-Bill rates past models based on the hypothesis that these rates follow a martingale process: Past returns contain no information usable for detecting future returns.

There are several reasons why it makes sense to consider nonlinear dynamics and mathematical chaos in financial environments:

1) You can amalgamate technical and fundamental analyses of rates/prices. Sometimes the factors belonging to fundamental models have a bigger influence; sometimes the technical aspects dominate.

2) Chaotic nonlinear systems provide behavior that often appears random, but actually has a "hard-to-detect" order. In many models that treat certain variables as random, nonlinear dynamics could explain this randomness just as chaotic behavior.

3) Recent investment strategies all depend on market-momentum and short-term-prediction. These are significant properties of nonlinear dynamical systems. So there is a good chance that at least some parts of the market behave like a nonlinear dynamical system.

4) Even if nonlinear dynamics cannot provide long-term forecasting, it can describe the future evolution of disorder. So we could probably make predictions of how chaotic certain markets behave.

The aim of the article is to find a model for the Treasure-Bill stock rates that contains both, fundamental and technical approach. Furthermore the authors examine, in which cases the fundamental "part" becomes superior and in which cases the technical influences overbalance.

The Model

The originator of this approach is Jensen and Urban's work, "Chaotic Price Behavior in a Nonlinear Cobweb Model", Economic Letters, 1986. Jensen and Urban present a modified version of the nonlinear model (2):

$r_{t+1} = a - b\, r_{t}, + c\, r_{t}^{2}$ (3)

The authors point out that this simplification of equation (2) can be made without loss of generalization, because recent mathematical results indicate the insensitiveness to the details of the nonlinear terms in equation (2).
Equation (3) shows similar behavior as the logistic map and indicates chaotic behavior of the rates of Treasure-Bills, which formerly was seen as randomly.

The model of Maurice Larrain merges a nonlinear component, which he calls "K-map" (and which represents the technical equation) with a behavioral component, which he calls the "Z-map" and which represents the fundamental equation:

$r_{t+1} = f(r_{t}^{\,n}) + g(Z_{t,\dots,t-i}^{\,n})$ (4)

Here, f represents the K-map, whereby g represents the Z-map. n is the degree of nonlinearity of r. "t,...,t-i" insinuates that relevant data for the rate of time t is given by the times t,...,t-i: Between cause and effect there can be lags.
There is a well-known approximation of the map. Merging this map with the map of equation (3) you get the following:

$r_{t+1} = [a + b(r_{t}^{\,n}) - c(r_{t}^{\,n+1})] + [d(y)_{t} + e(P)_{t} - f(MS)_{t} - g\Sigma(Y\text{-}C)_{t}]$ (5)

The variables have the following meaning:

y	real GNP (Gross National Product)
P	Consumer Price Index
MS	nominal money supply M2
Y	personal income
C	real personal consumption
Y-C	savings
Σ(Y-C)	wealth

Experimental Results

The best results you get with n=2, which are given in the following table:

Period	No.	R^2	DW	Const	$r^2(t)$	$r^3(t)$	GNP(t)	P(t-l)	MS(t-2)	W(t-2)
Constant Number of Observations										
62Q1 81Q1	77	.9231	1.66	1.8	1.4	7.7	7.6	3.1	6.8	13.4
				(-5.4)	(9.0)	(-6.7)	(5.7)	(4.3)	(-1.9)	(-6.0)
63Q1 82Q1	77	.9333	1.66	1.8	1.3	6.5	7.7	2.7	5.9	12.8
				(-4.8)	(9.0)	(-7.1)	(5.1)	(3.8)	(-1.6)	(-5.4)
64Q1 83Q1	77	.9072	1.54	1.8	1.1	5.1	9.2	1.9	7 5	9.6
				(-4.0)	(7.0)	(-4.9)	(4.8)	(2.4)	(-1.7)	(-3.7)
65Q1 84Q1	77	.9082	1.56	2.3	1.2	5.5	11.4	2.5	12.4	9.5
				(-4.8)	(7.5)	(-5.4)	(5.5)	(3.4)	(-3.1)	(-4.1)
66Ql-85Q1	77	.9099	1.55	3.0	1.3	6.2	14.1	3.3	17.8	9.7
				(-5.9)	(8.2)	(-6.2)	(6.4)	(4.8)	(-5.2)	(-4.5)
67Q1 86Q1	77	.9121	1.63	3.2	1.3	6.3	15.0	3.4	18.9	9.3
				(-6.4)	(8.9)	(-6.8)	(7.0)	(5.4)	(-6.2)	(-4.5)
68Q1 87Q1	77	.8983	1.61	2.6	1.2	5.4	12.6	2.4	13.5	7.7
				(-5.4)	(8.0)	(-5.9)	(6.0)	(4.3)	(-5.4)	(-3.6)
69Ql-88Q1	77	.8900	1.62	2.3	1.2	5.2	11.3	1.8	10.7	6.5
				(-4.8)	(7.7)	(-5.5)	(5.4)	(3.4)	(-4.7)	(2.8)
70Ql-89Q1	77	.8937	1.68	2.4	1.1	5.0	11.4	1.6	10.8	5.4
				(-5.1)	(7.6)	(-5.4)	(5.7)	(3.2)	(-4.9)	(-2.3)
Increasing Number of Observations										
65Ql-81Q1	65	.9050	1.70	2.2	1.4	8.0	9.4	3.4	9.7	13.7
				(-5.1)	(8.5)	(-6.4)	(5.1)	(4.3)	(-2.3)	(-5.7)
65Q1 82Q1	69	.9265	1.68	2.2	1.3	6.7	9.7	3.1	9.1	13.1
				(-4.8)	(8.8)	(-7.0)	(5.0)	(4.0)	(-2.1)	(-5.3)
65Q1 83Q1	73	.9054	1.55	1.3	1.2	5.5	11.2	2.4	11.2	- 10.0
				(-4.5)	(7.2)	(-5.2)	(5.2)	(2.9)	(-2.3)	(-3.8)
65Ql-84Q1	77	.9082	1.56	2.3	1.2	5.5	11.4	2.5	12.4	9.5
				(-4.8)	(7.5)	(-5.4)	(5.5)	(3.4)	(-3.1)	(-4.1)
65Q1 85Q1	81	.9078	1.50	2.5	1.2	5.7	11.8	2.9	14.7	9.0
				(-5.1)	(7.7)	(-5.6)	(5.7)	(4.2)	(-4.4)	(-4.2)
65Ql-86Q1	85	.9070	1.50	2.4	1.2	5.6	11.7	2.9	14.7	8.9
				(-5.2)	(8.2)	(-6.0)	(5.8)	(4.5)	(-5.0)	(-4.3)
65Ql-87Q1	89	.9029	1.53	2.1	1.1	5.0	11.3	2.2	11.2	8.0
				(-4.8)	(7.9)	(-5.6)	(5.5)	(4.1)	(-4.8)	(-3.9)
65Ql-88Q1	93	.8990	1.55	1.8	1.1	4.8	9.4	1.7	9.0	-7.4
				(-4.4)	(7.8)	(-5.4)	(5.1)	(3.6)	(-4.4)	(-3.6)
65Ql-89Q1	97	.8990	1.55	1.9	1.1	4.8	9.6	1.7	8.8	7.7
				(-4.7)	(8.1)	(-5.6)	(5.4)	(3.6)	(-4.4)	(-4.0)

No. = number of observations; R^2 = R-squared; t-values in parentheses; DW = Durbin Watson statistic; $r^2(t)$ = first nonlinear term; $r^3(t)$ = second nonlinear term; GNP(t) = real GNP; P(t-1) = Consumer Price Index; MS(t-2) = money supply (M2); W(t-2) = real wealth. The dependent variable is a two quarter moving average of r(t + 1), the 91-day T-bill rate.

In the table, quarterly averaged data for the Treasury-bill rate is analyzed over the period 1962-1989. t is a discrete time-parameter that moves over the quarters.
Using the rolling regression method (constant period moved over the time) and growing regression

method (increasing the number of observations) you get several repetitions of the same experiment, which allows you to make statements about the quality of the model.

Each of the first nine rows shows characteristic values of the rolling regression method; each of the last nine rows describes a regression made with the growing regression method.
In the following the meaning of the characteristic values is described:

Period	Period over which the regression method was taken.
No.	Number of observations (quarters).
R^2	Coefficient of determination; it ranges between 0 and 1 and describes the percentage variation in interest rates explained by the variables of equation (5).
DW	Durbin-Watson statistic. This index says something about the randomness of the error terms of the regression equation. A DW of 2.0 is optimal, lower values indicate a positive bias, higher values a negative one. Values between 1.6 and 2.4 are satisfactory. You cannot use the DW for determination of the randomness of the error terms if the current interest rates are linearly dependent on past interest rates. But here we have nonlinear dependence.

The description of the other parameters is given above, below equation (5).
The Consumer Price Index P is lagged one quarter what is indicated by subtracting 1 from the current time t. Similarly, the nominal money supply MS and the real wealth are lagged two quarters. Values without a parenthesis describe the weight of these variables, values in parenthesis give the so called "t-values".
The t-values measure the stability and statistically significance of a parameter. Negative values show inverse relation to the interest rates, that is, a rise in these values lower the rates. Values of absolute greater 1.96 are satisfactorily.
For example, take row 1:
The two quarters lagged money supply MS has a weight of -6.8 and a t-value of -1.9. The one-quarter lagged Consumer Price Index P has a weight of 3.1 and a t-value of 4.3.

Interpretations

From Equation (5) it follows that the nonlinear part of the map, the K-map, provides acceleration forces with the term $b(r_t^n)$ and deceleration forces with the term $-c(r_t^{n+1})$ on the interest rates.
For example, $b(r_t^n)$ could be interpreted as corrections to volatile market overreactions. The term $-c(r_t^{n+1})$ could for example stand for bandwagon effects when the rates are rising.
The table shows that the coefficients of the Z-map, d, e, f and g, vary more than the coefficients of the K-map, a and b, over the time. Larrain draws the conclusion that this shows that the completion of technical behavior to the fundamental model may be appropriate.

In the following, the behavior of the technical part of equation (5) is described.

Description of the Nonlinear Dynamics in the Technical Model

For a description of the nonlinear part of equation (5) we have to make some simplifications:

First, let us assume that the fundamental economical variables, used in the Z-function, are constant Z.
Further we assume that parameter a is zero and the parameters b and c are equal.
Then we get the following equation out of (5):

$r_{t+1} = r_t[c(1 - r_t)] + Z$ (5a)

As we see, the logistic function is "embedded" in equation (5) and this gives first hints on the (partly) chaotic character of (5).
The author was by this encouraged to examine the chaotic behavior of the map. For this attitude he assumed that b and c are not equal and c is constant:

$r_{t+1} = b\, r_t^2 - c\, r_t^3 + Z$ (5b)

For this function we get under variation of the b-parameter between 1 and 1.8 the following bifurcation diagram. Unfortunately the author does not state which values for c and Z and starting points for the iteration he used. Although I tried several values, it was nor possible to reproduce his diagrams:

For values of b over 1.68 the bifurcation diagram exhibits chaotic behavior.
Larrain also gives a picture of the Lyapunov Exponent of this function, which is shown below.
The Lyapunov-Exponent function diagram supports the statement of chaotic disorder.
For values of b bigger then 1.68 we find a positive Lyapunov-Exponent, what shows that nearby points diverge under iteration:

We find that for certain values of Z, c, b and a the model (5) provides similar behavior like the logistic map.
The model turns for values of b between 1.5 and 1.68 from order to chaos.
Important is that such chaotic behavior is nearly not distinguishable from a random process.

Advantages of the New Model

If a (partly) chaotic behaving model is almost not distinguishable to a (partly) random model, then the following question arises:
What benefits we can draw out of our nice new nonlinear dynamic model?
Here are some arguments that have to be mentioned:

- As we see from the bifurcation diagram the change from order to chaos occurs in the typical manner of dynamical systems, the period doubling.
  If we consider T_n as the value of coefficient b at which the period doubles for the nth time, then the quotient $d_n = (T_{n+1} - T_n)/(T_{n+2} - T_{n+1})$ tends for n to infinity independent of the system nature to Feigenbaum's constant 4.6692016.
  So perhaps we will not be able to make exact predictions about the behavior of the interest rates but we may are able answer questions of the following nature:

  - Are we in the moment in the "stable" environment such that short-time predications are possible?

- How fast is the system going to change from a "stable" environment to an unstable, chaotic one?

- Let us for a moment remember the bifurcation diagram of the logistic map. As insinuated by the picture below, we find everywhere in the bifurcation diagram self similarity (the box in the left corner is a magnification of the small box between the values 2.8 and 2.9).
  In particular we find (small) areas that are in a chaotic neighborhood and where short-time prediction is possible - for example for values inside the white slice at around 2.85.

- A last argument is given by the set of the dark streaks that we see over the whole domain of chaotic behavior.
  These streaks, Larrain calls them "probability streaks", suggest values that somehow "attract" iterations - they occur more frequently then other values.
  By this we can make statistical / probabilistical statements about prospective trends in our model.

The interaction between K-map and Z-map

The two components of the model, the K-map and the Z-map determine the development of the Treasure-Bill rates. An important question is:
When is the influence of the K-map overriding the Z-component and when is it just the other way around?
Let us start with the question which market situations are simulated by the K-map.
Larrain mentions market activities related to profit-taking, bandwagon effects and market panic.
This and the chaotic behavior of the K-map lead to the conjecture that a more erratic interest rate shows an overriding K-map.
The K-map may determine the rate when most of the participants in the market do not believe in the economic variables anymore. The causes for such a mistrust in the economic data could be caused by exceptional environments like a war threat, mistrust in the government but also a steadily growing suspiciousness to financial models.

Final Remarks

It is worth to point out that an overriding K-map is a necessary but not a sufficient condition for chaotic behavior in the interest rates.
Only under special conditions the K-map shows such behavior. The parameters a, b, c and Z have to be chosen in an appropriate way to get a bifurcation diagram like the one above. With another choice of values there may be attracting fixedpoints for the K-map.
Furthermore, it is not clear, which effects are given by the simplification of taking the Z-function as constant.
The merging of the Z-function with the K-function to equation (8) may not cause chaos at all. In fact, the coefficients a, b, c and Z, which guide to the bifurcation diagram above, were not found by the tests of the quarterly data.

Nevertheless equation (5) gives doubtless a very good approximation of the real market and provides an explanation for seemingly random influences on the market.
Anyway, by the results of Larrain the hypothesis that nonlinear chaotic dynamics influences the financial market cannot be rejected.

Assuming that nonlinear dynamics de facto influences the interest rates, Larrain gives some tentative propositions:

1) For shorter time periods of analysis the influence of the K-map will grow and the influence of the Z-map will fade.

2) For certain values of the coefficients of the K-map erratic behavior will overtake a financial market, if the influence of the fundamental parameters is negligible.

3) The market *can* suspend belief in the Z-map in certain times. This would cause a domination by the K-map.

Whether the thesis of a chaotic influence of nonlinear dynamical functions on the financial market rather than a random influence according to the random-walk hypotheses is true or not has to be figured out in carefully statistical studies on the market. Larrain's thesis offers the opportunity to a deeper understanding of the processes behind the seemingly randomness of the financial market.